On the Approximability of Weighted Model Integration over DNF Structures [1]

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09 July 2020

MCW@SAT2020

Weighted Model Counting

ϕ propositional formula

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$w: \mathcal{A}(\phi) \rightarrow \mathbb{R}$ weight function

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$w: \mathcal{A}(\phi) \to \mathbb{R}$ weight function

$$\mathrm{WMC}(\phi) \qquad \sum_{\nu \vDash \phi} w(\phi)$$

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Applications

Probabilistic Graphical Models

Probabilistic Logic Programming

Probabilistic Databases

Probabilistic Knowledge Bases

Weighted Model Integration

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$\mathbf{X}$ set of n real variables

$\mathbf{V}$ set of m Boolean variables

Weighted Model Integration

$c_1 x_1 + \ldots + c_i\, x_i \bowtie c$

LRA Atom, $x_i \in \mathrm{X}$, $\bowtie \in \{<, \leq, >, \geq, =, \neq\}$

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$\mathrm{WMI}(\phi)$

$$\sum_{v} \int_{x_\phi} w(x, v)\, dx$$

where v is a Boolean assignment over V, x_ϕ denotes valuations of X satisfying ϕ.

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For weight functions in WMI, it is common to factorize [2] w as a product of m Boolean literal weights and a density function over real variables, i.e.,:

$$w(x, v) = w_x(x) \prod_{i=1}^{m} w_b(p_i).$$

Special Cases

$(x \vee y) \wedge (y \vee z)$

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➡ Special cases WMI(CNF), WMI(DNF)

Both WMI and WMC are #P-hard for exact solving. Hence, we study WMI within the context of approximate solving.

Approximation Hardness

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	WMC		WMI
CNF	NP-hard [3]	→	NP-hard
DNF	FPRAS [4]	→	?

Result:

Show that WMI(DNF) admits an FPRAS for concave weight functions

Result builds on existing FPRAS algorithms for WMC(DNF) and volume computation for the union of convex bodies

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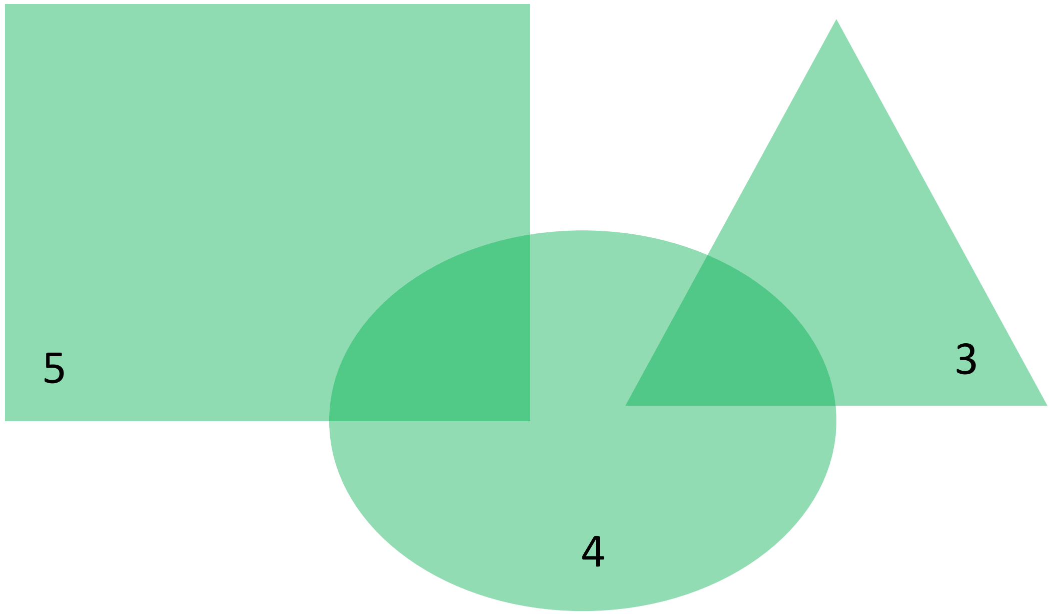

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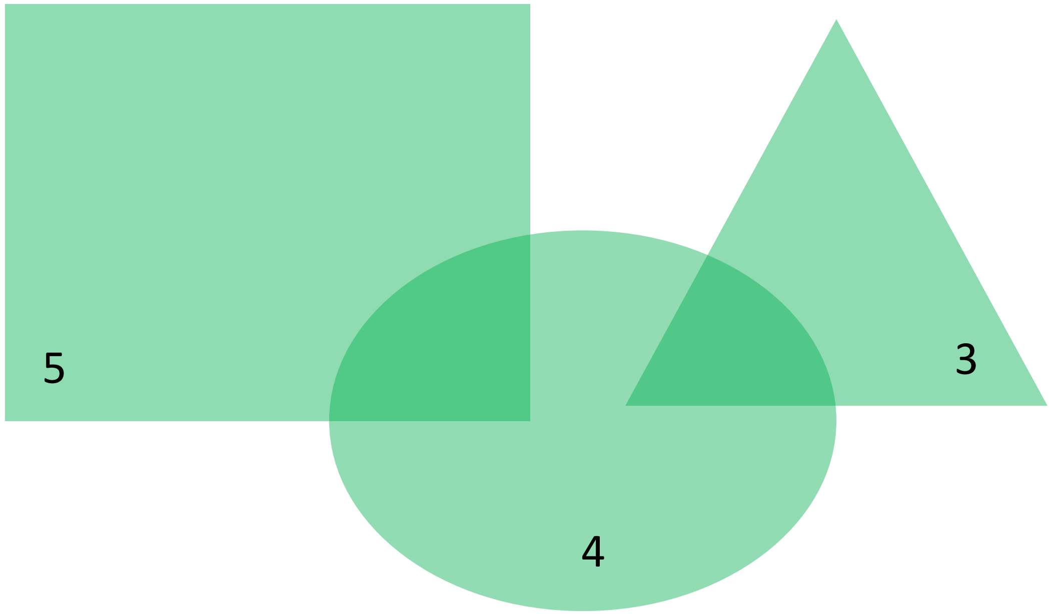

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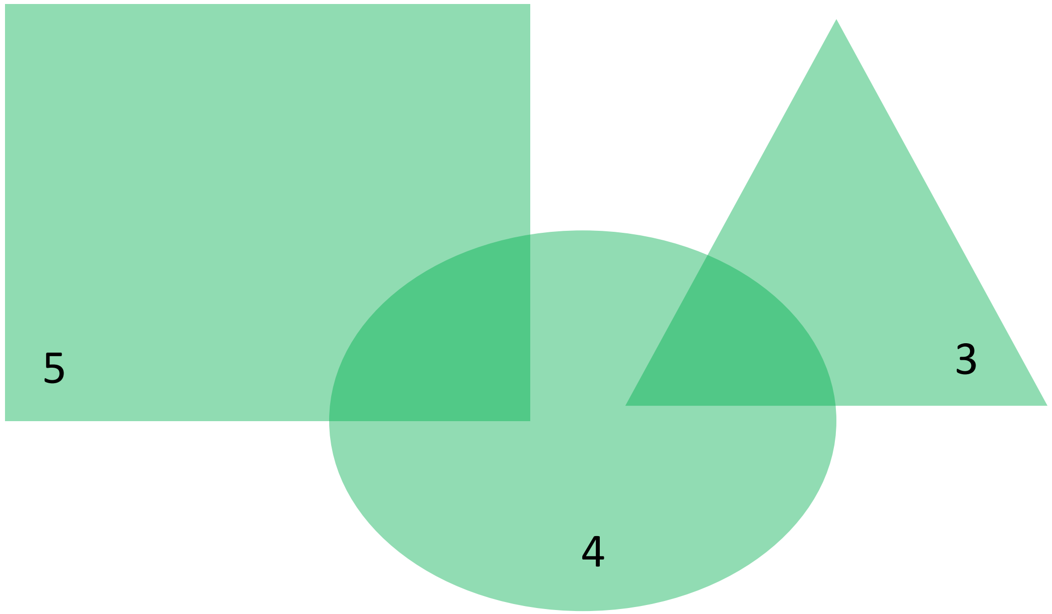

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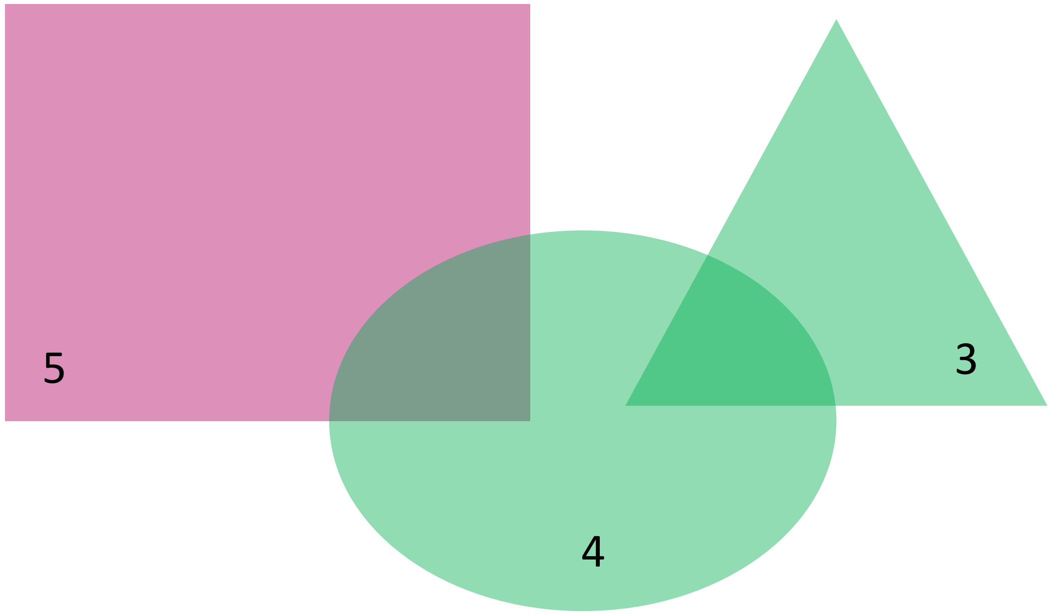

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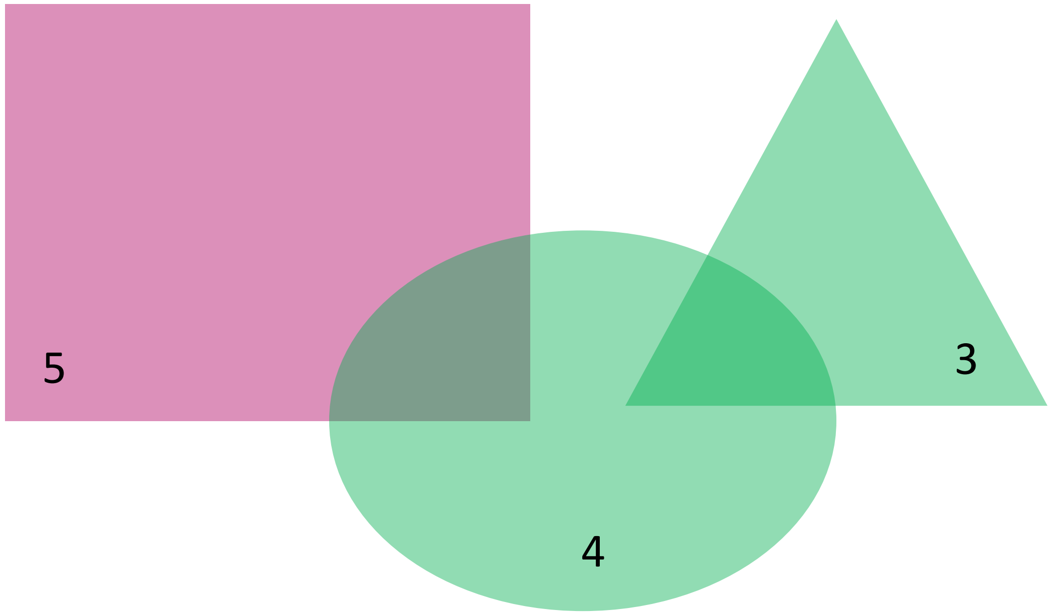

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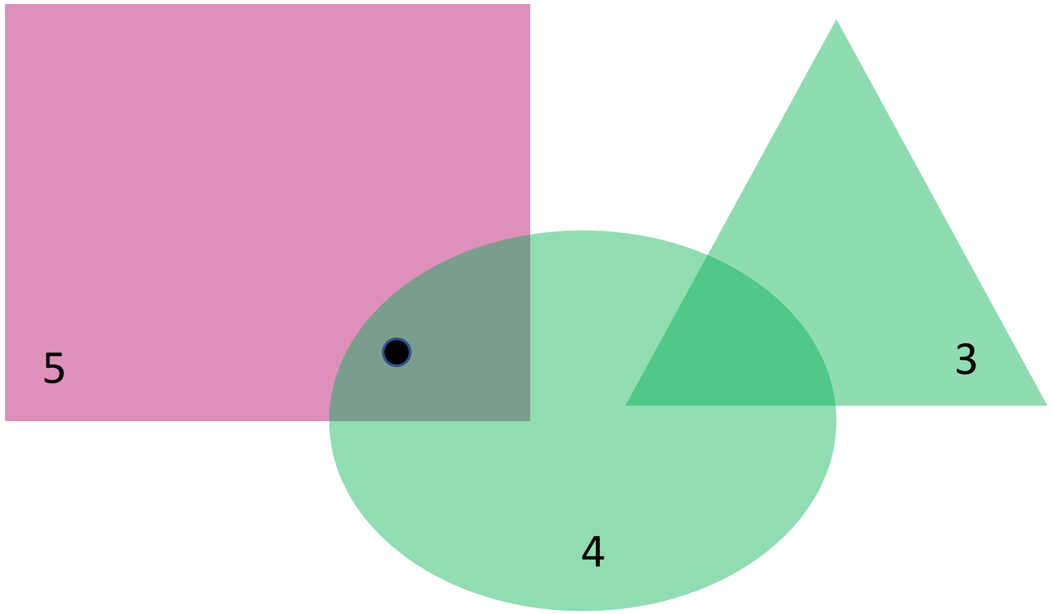

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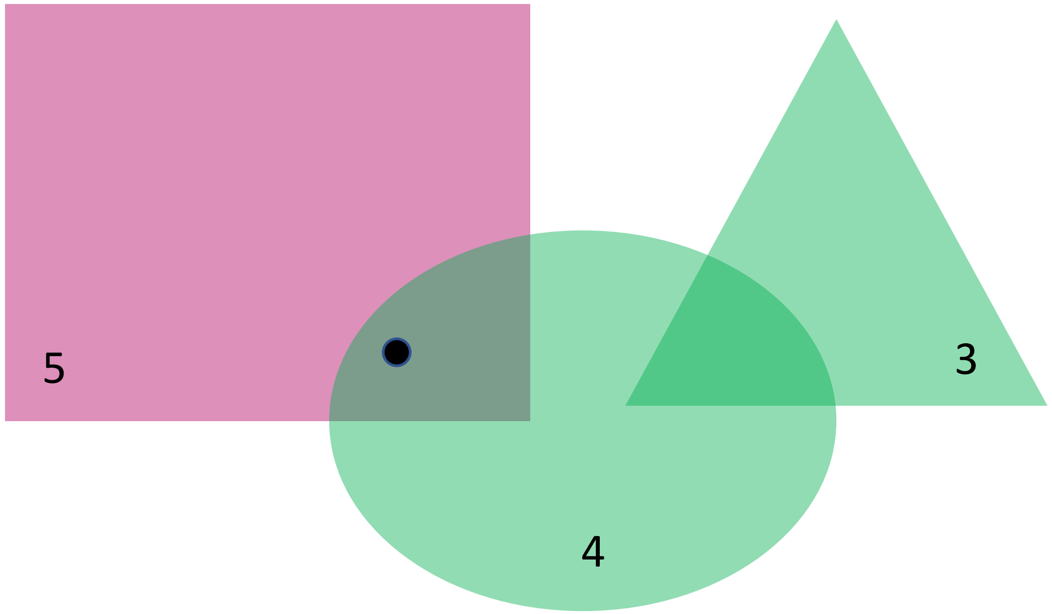

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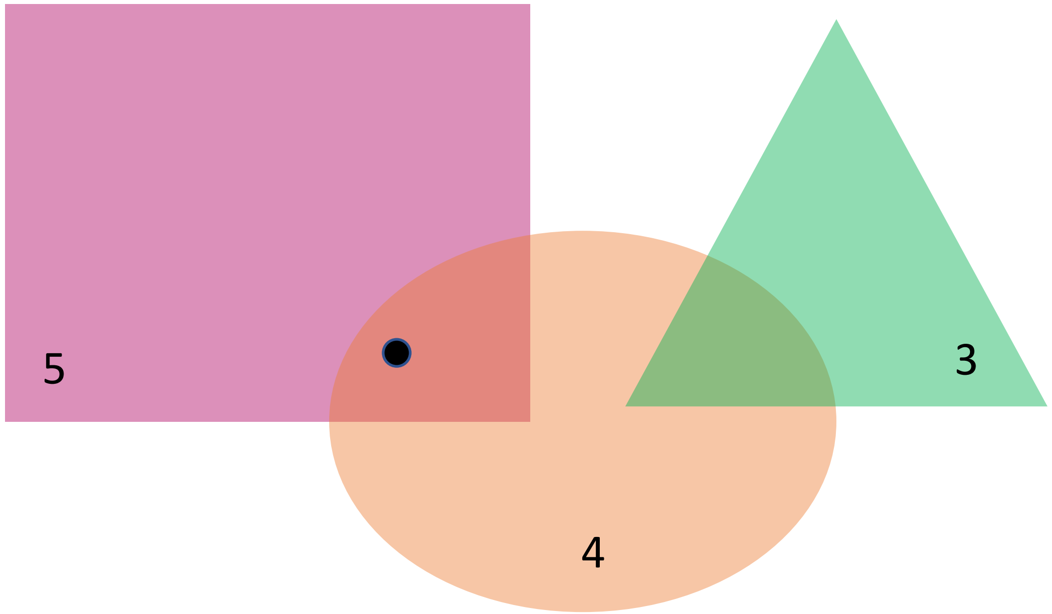

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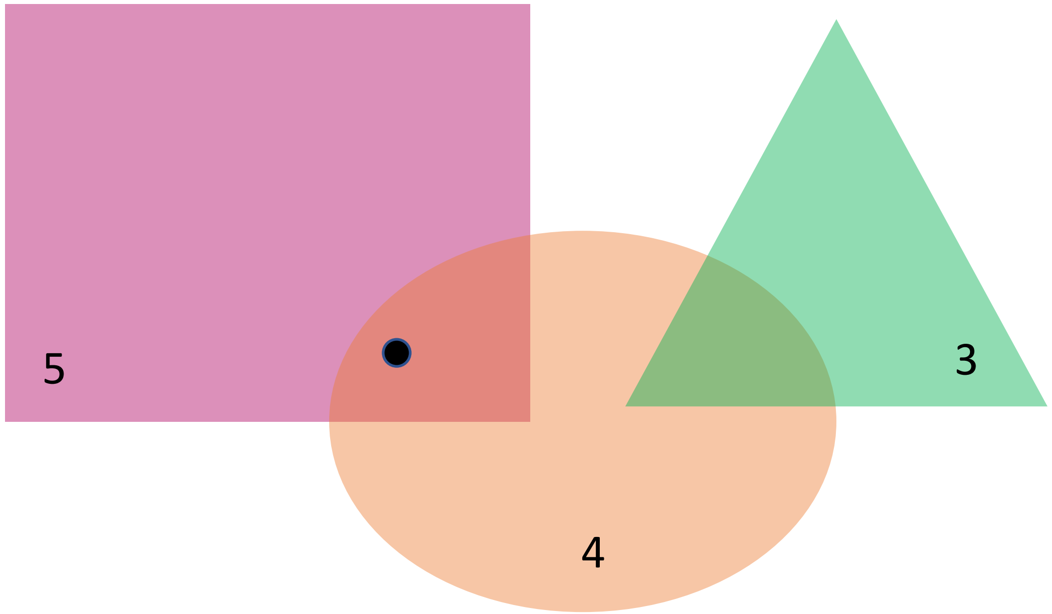

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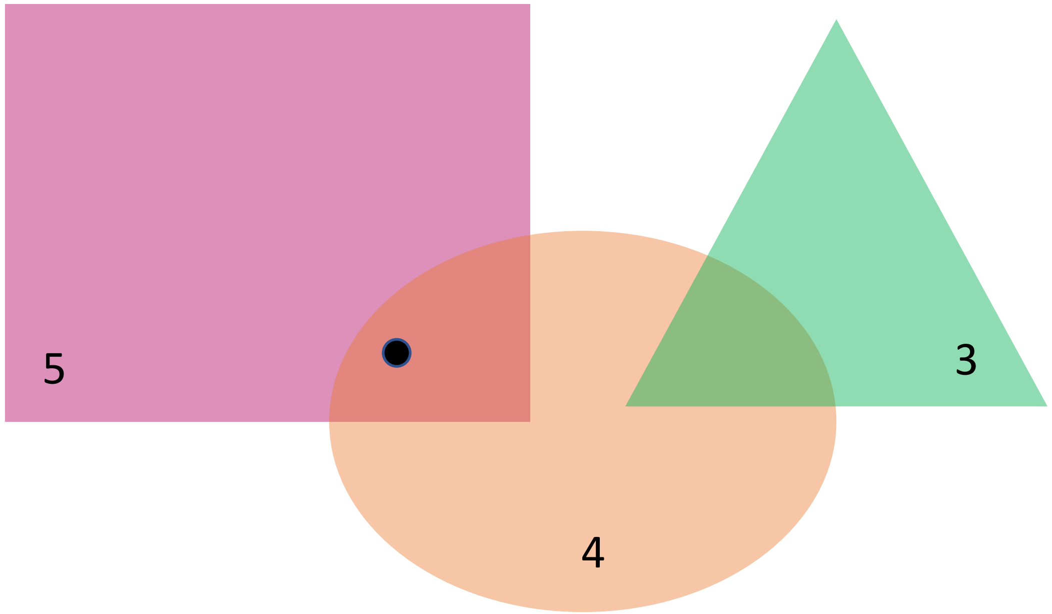

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Number of successes then yields an unbiased estimator for volume

ApproxUnion is an FPRAS

1. Volume Computation: FPRAS with error ϵ_V, confidence δ_V

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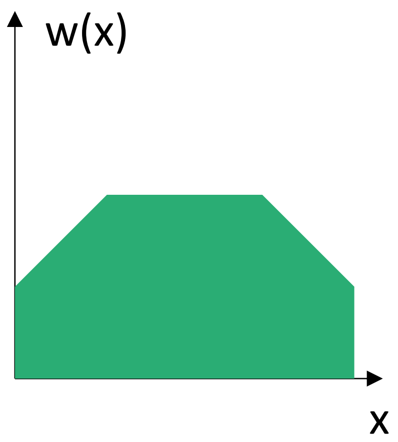

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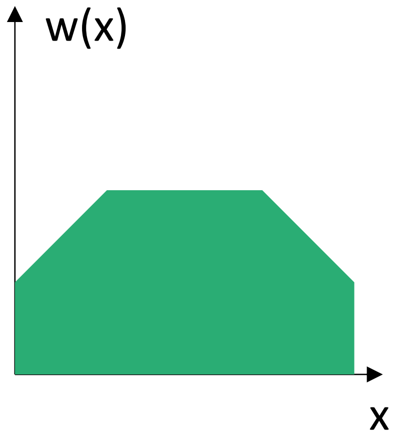

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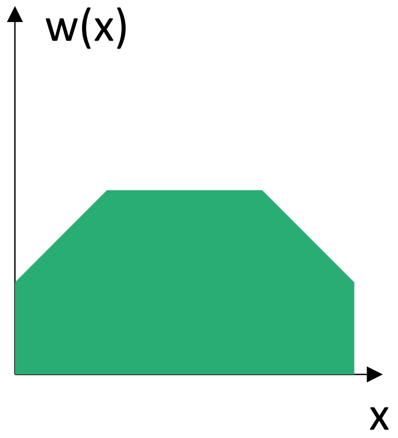

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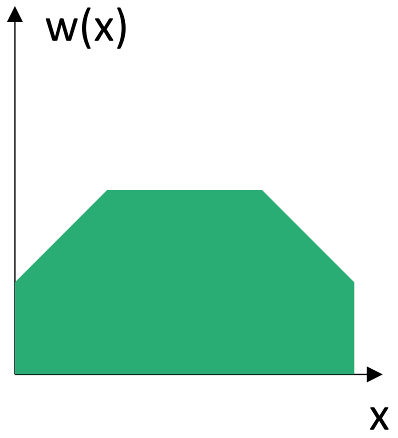

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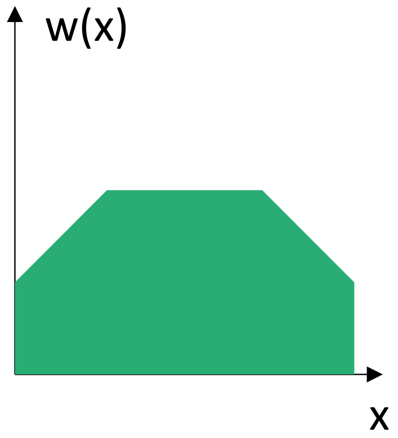

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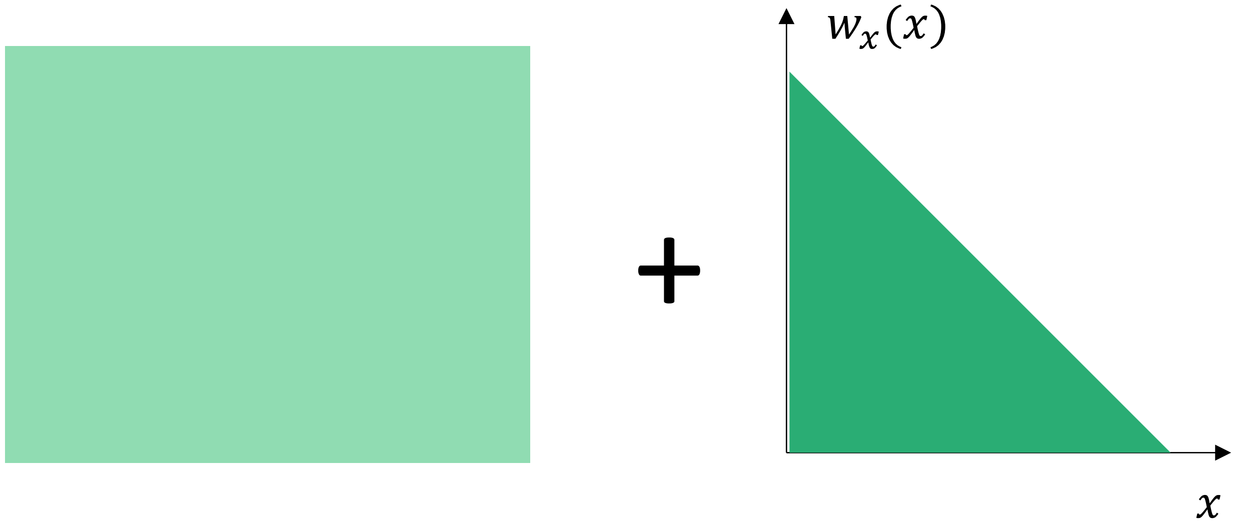

2D Convex Polytope

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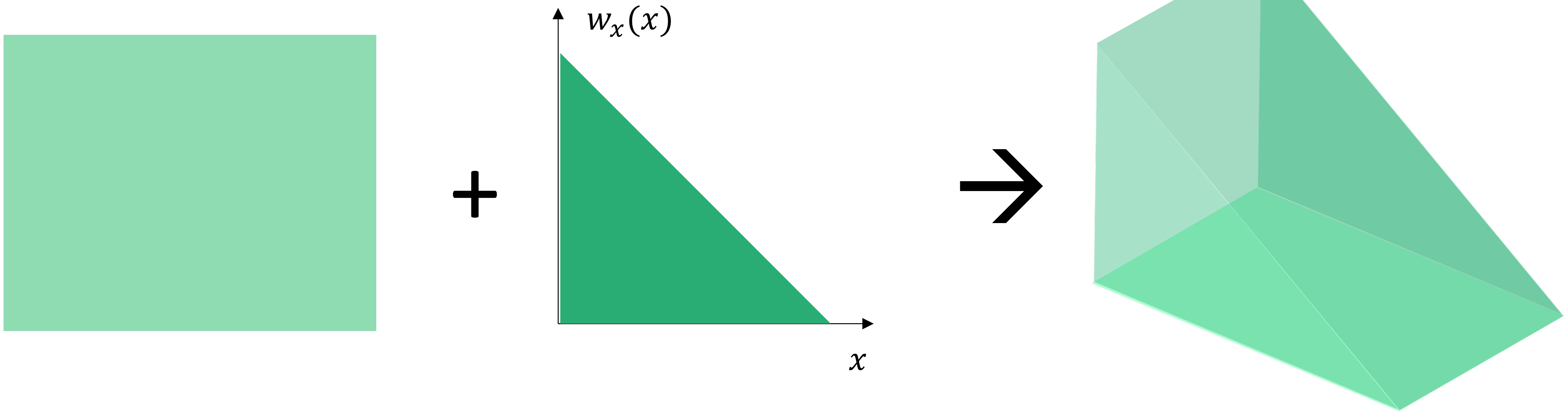

2D Convex Polytope | Concave Weight Function | 3D Convex Polytope

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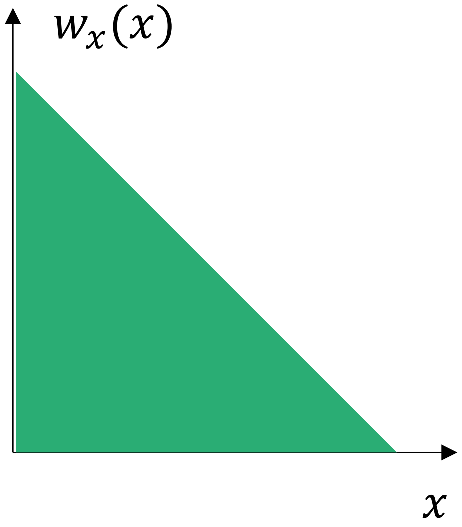

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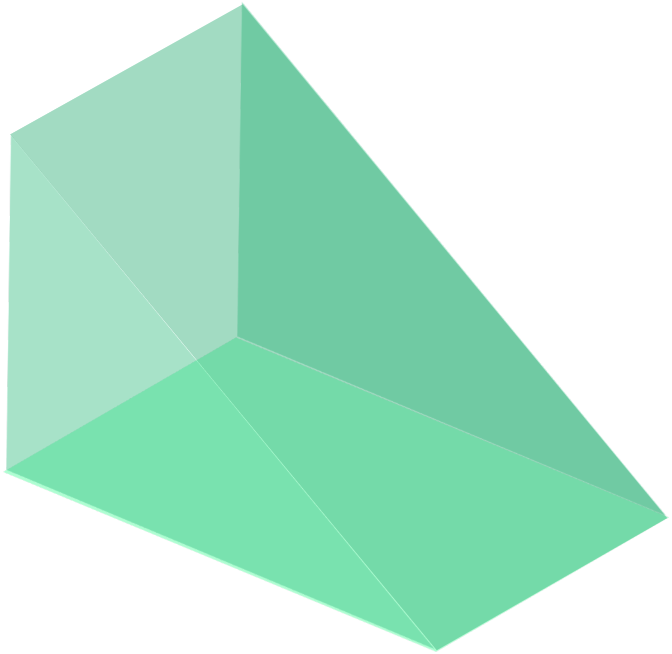

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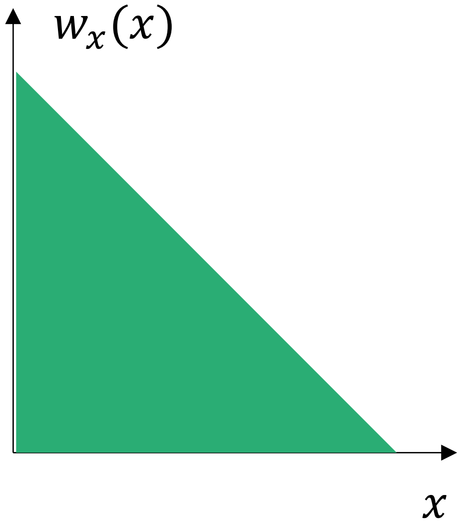

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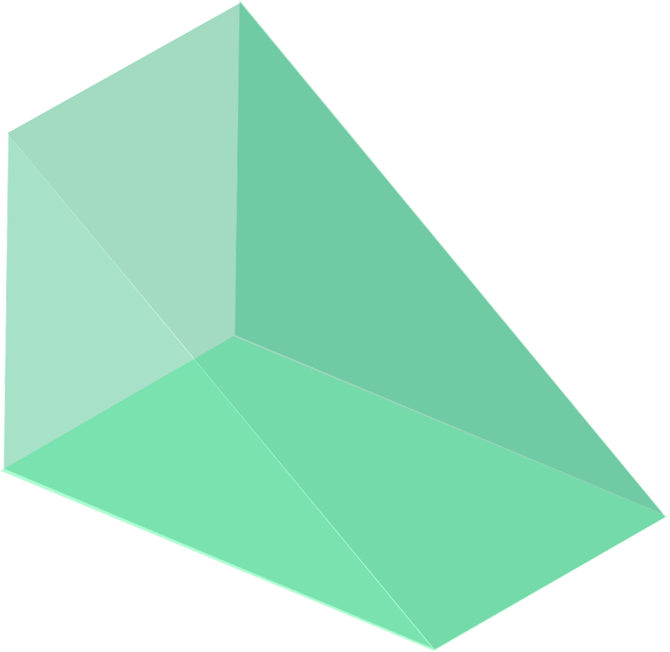

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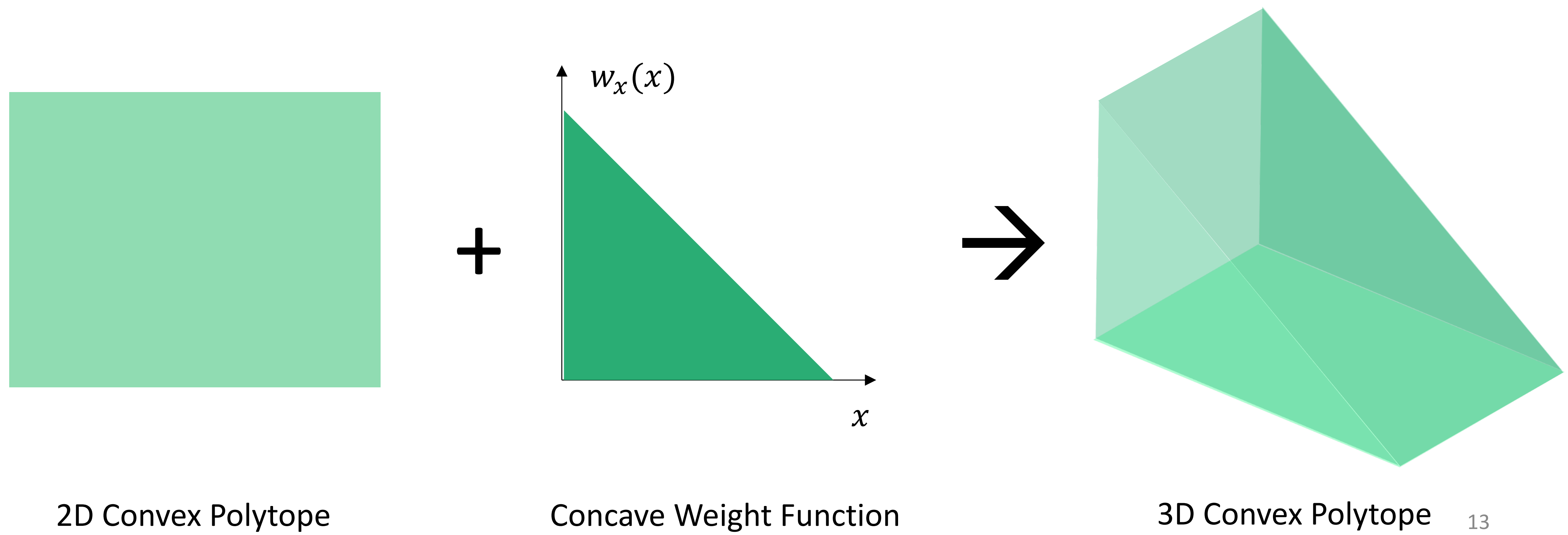
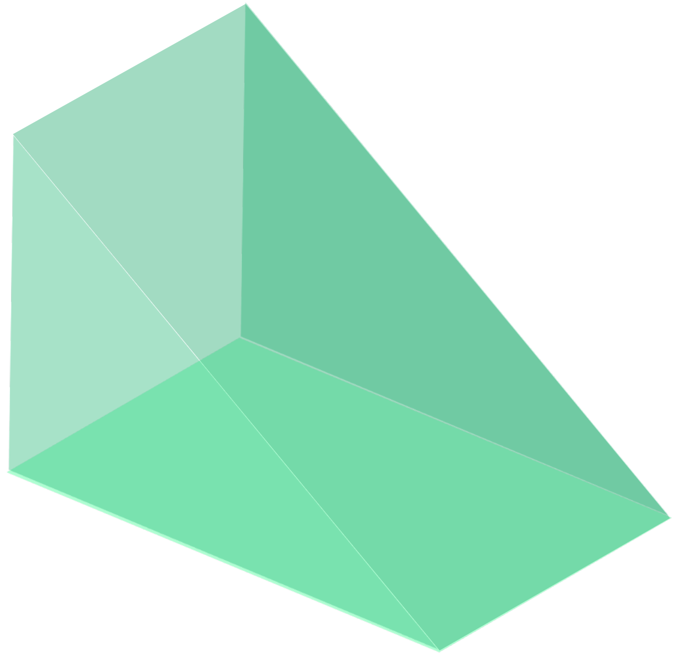
- This volume is multiplied by Boolean probabilities to return clause weight.



ApproxWMI: Sampling Points and Checking Membership

- The *n+1*-dimensional polytope is sampled uniformly, as a proxy for sampling the original polytope according to the weight function.



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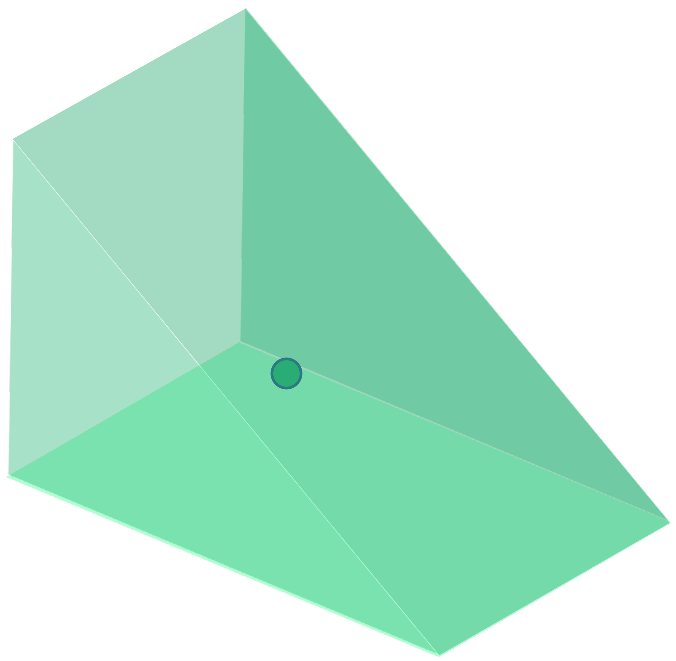

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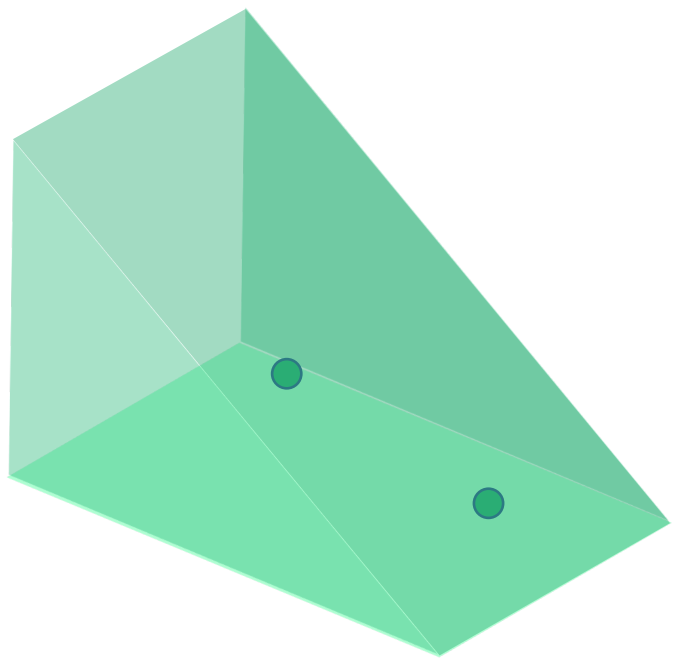

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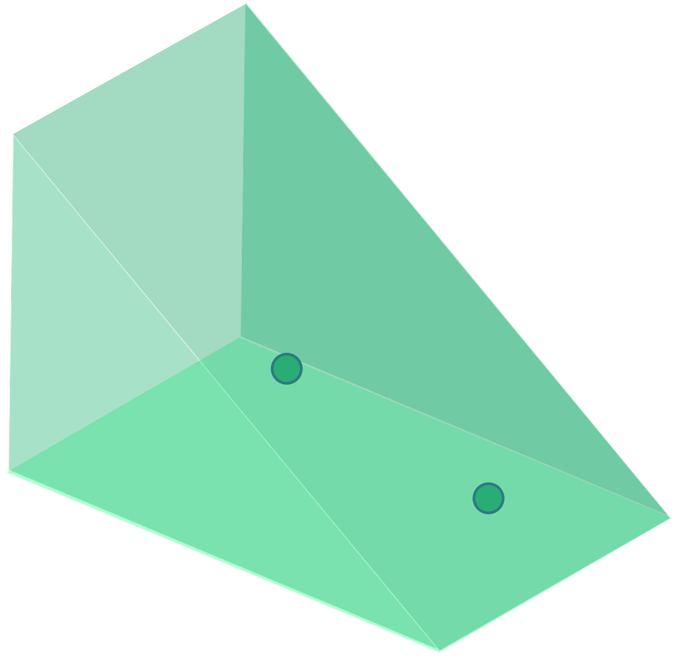

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- Real weight function is a concave polynomial with degree up to 5.

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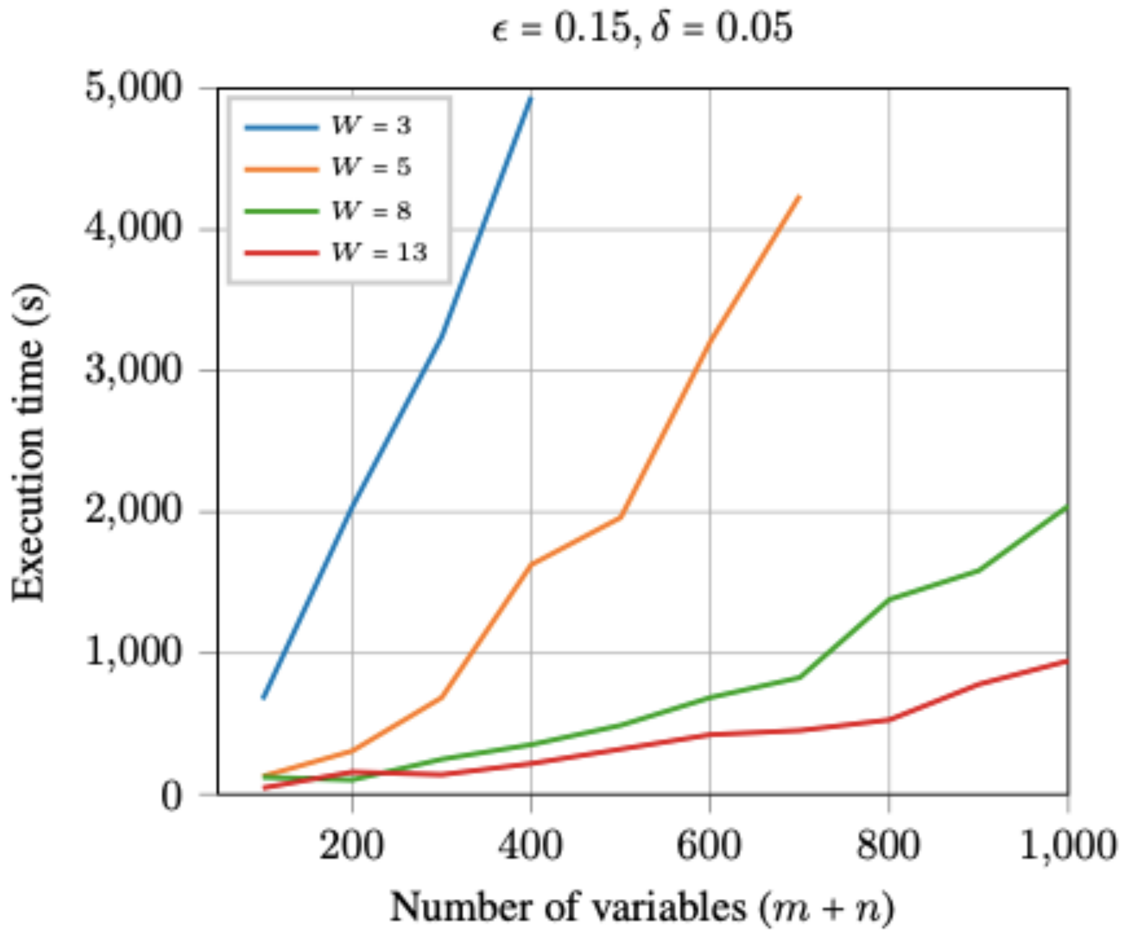
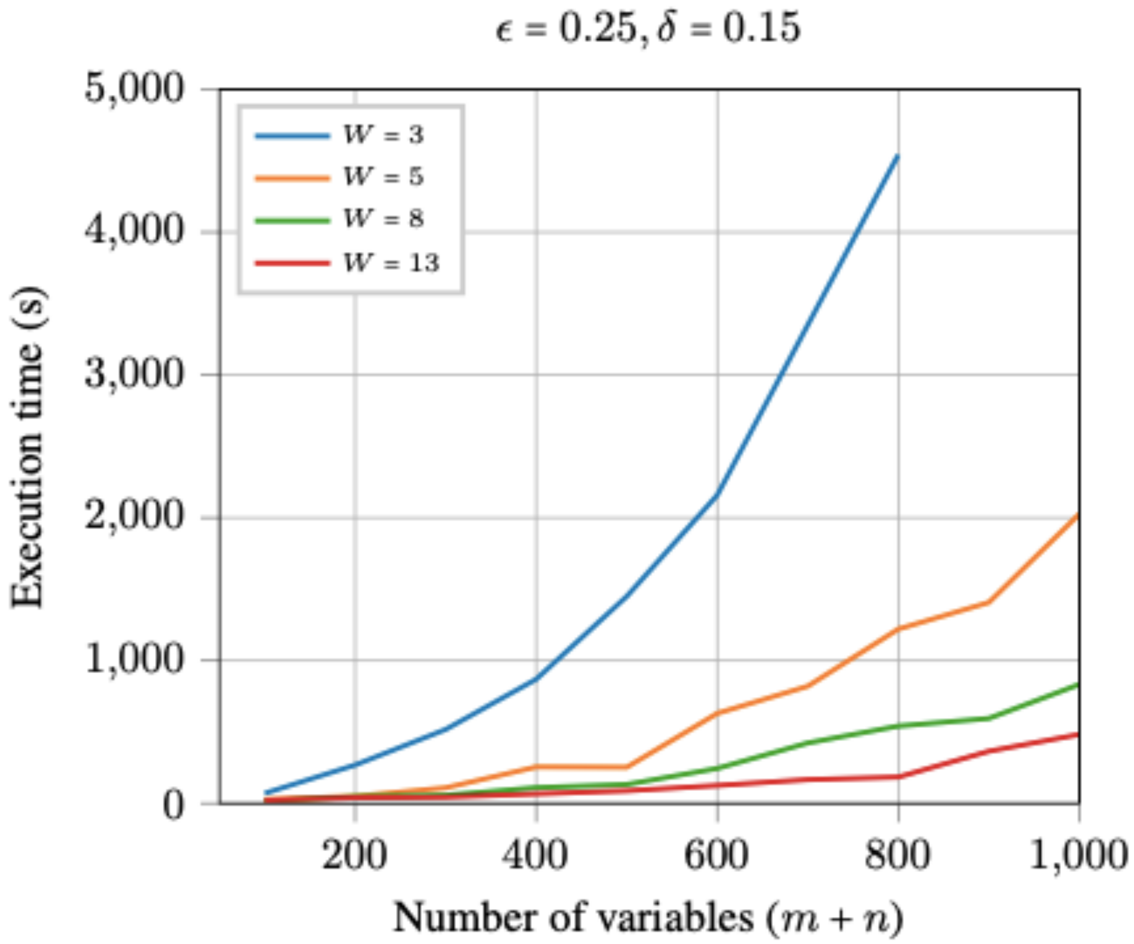
- Target confidence δ: Set to 0.05, 0.15 and 0.25.

Results



$\epsilon = 0.15, \delta = 0.05$
5,000
4,000
3,000
2,000
1,000
0
Execution time (s)
$W = 3$
$W = 5$
$W = 8$
$W = 13$
200
400
600
800
1,000
Number of variables $(m + n)$

Results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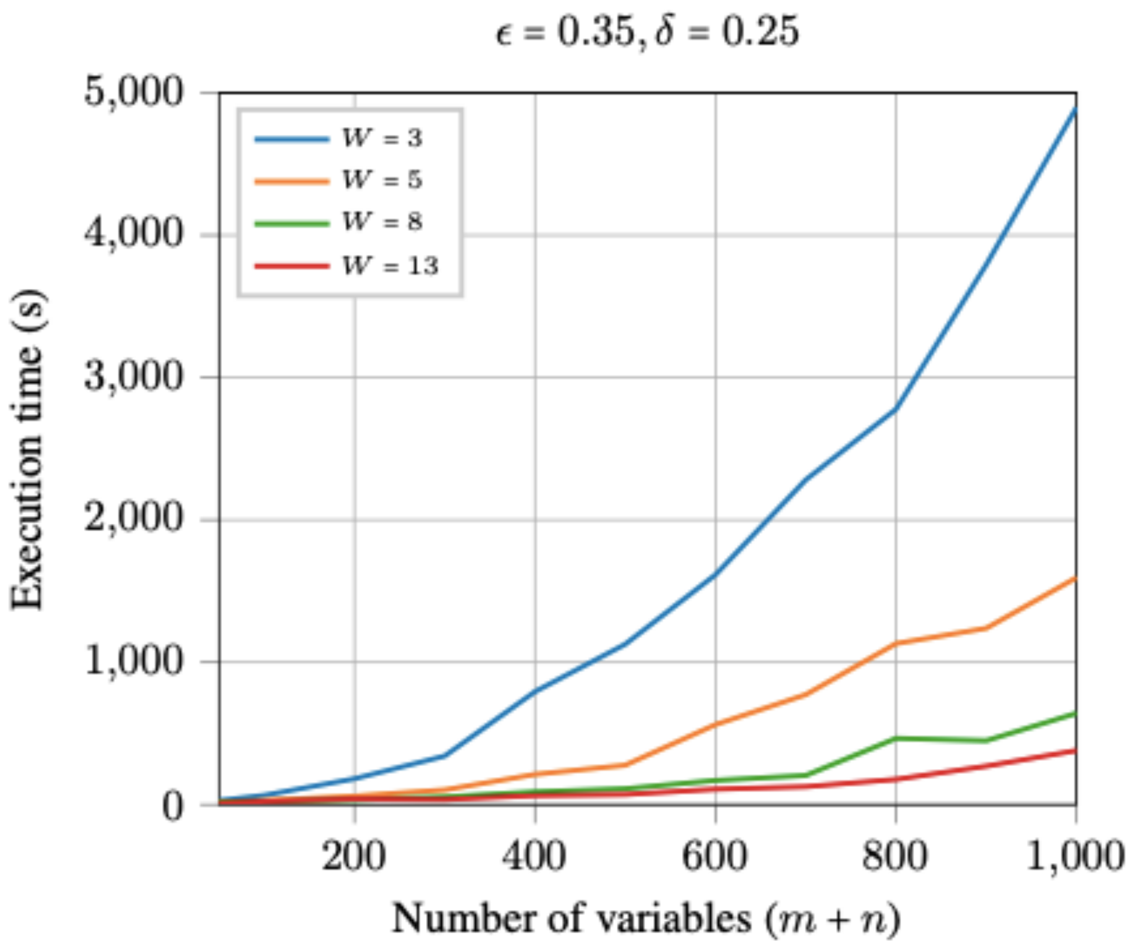
Number of variables (m + n)

Results


$\epsilon = 0.35, \delta = 0.25$
5,000
4,000
3,000
2,000
1,000
0
Execution time (s)
$W = 3$
$W = 5$
$W = 8$
$W = 13$
200
400
600
800
1,000
Number of variables $(m + n)$

Summary

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- ApproxWMI is scalable, and can efficiently solve instances with up to 1K variables, which is out of reach for existing WMI solvers.
- ApproxWMI can be extended to more general factorizations enabling real-Boolean dependency.
- ApproxWMI is a useful tool for efficient probabilistic inference in hybrid domains.

Thank You!

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